The (XOR-)Ising model and the Gaussian free field

Tomás Alcalde López
Based on joint works with Lorca Heeney, Marcin Lis and Avelio Sepúlveda
(arXiv:2602.05886 & arXiv:2602.06011)

EPFL — May 13

TU WIEN
FWF Austrian Science Fund

Outline

1. **Main Results – Continuum**
2. **The Models**
3. **The Coupling**
4. **Main Results – Scaling Limit**
5. **Conjectures – Ashkin-Teller and sine-Gordon**

Main Results – Continuum

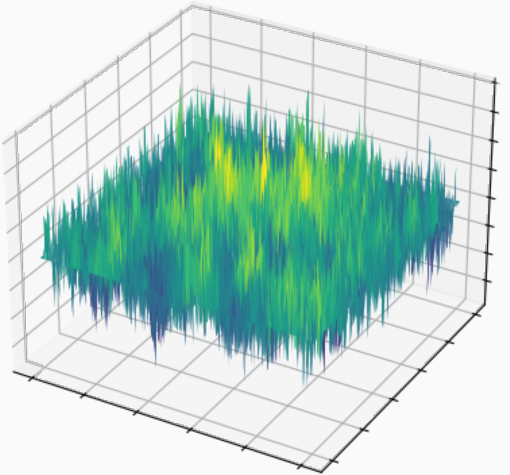
Gaussian free field

The Gaussian free field (GFF) in $D \subset \mathbb{C}$ is the centred Gaussian field h with

$$\mathbb{E}[h(z)h(w)] = G_D(z, w),$$

where G_D is the Green's function of $-\Delta$.

Guassian free field

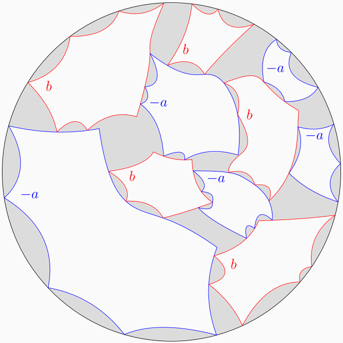


Two-valued sets of the GFF

For $a, b > 0$ with $a + b \geq 2\lambda$, the two-valued set $\mathbb{A}_{-a,b}$ is the connected component of

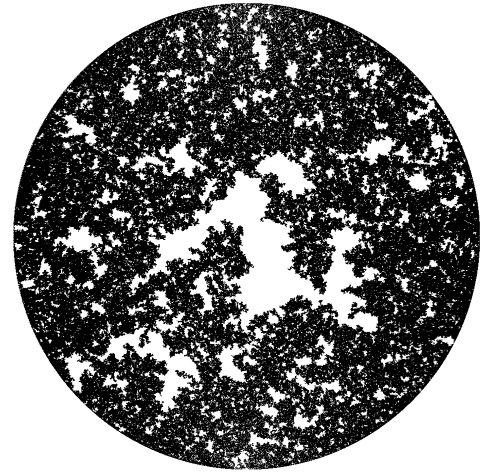
$$\partial D \cup \{z \in D : -a \leq h(z) \leq b\}.$$

Introduced by [Aru, Sepúlveda, Werner 2017].



Two-valued sets of the GFF

Note: The case $a = b = 2\lambda$ is the conformal loop ensemble CLE_4.



The coupling in one picture

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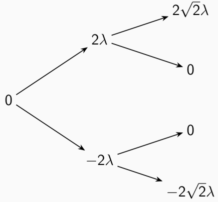

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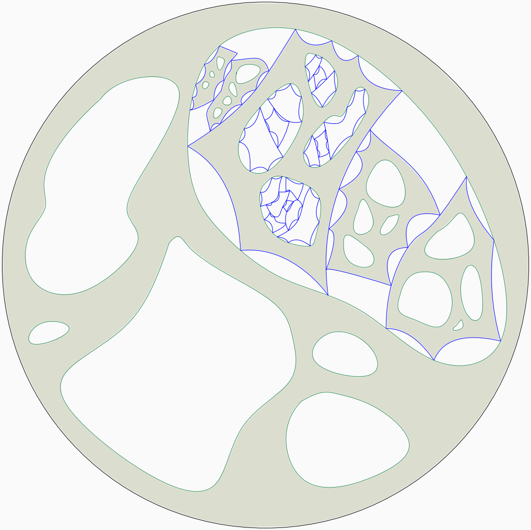
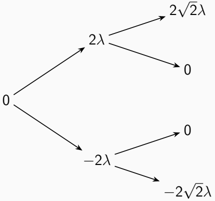


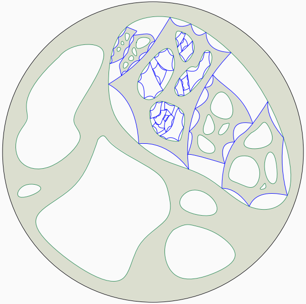


Ising decomposition

Theorem [AL, Heeney, Lis 2026]: The field

$$\sigma = \mu_0 + \sum_{k=1}^{\infty} \xi_k \mu_k$$

is the scaling limit of the critical Ising model with + boundary conditions.



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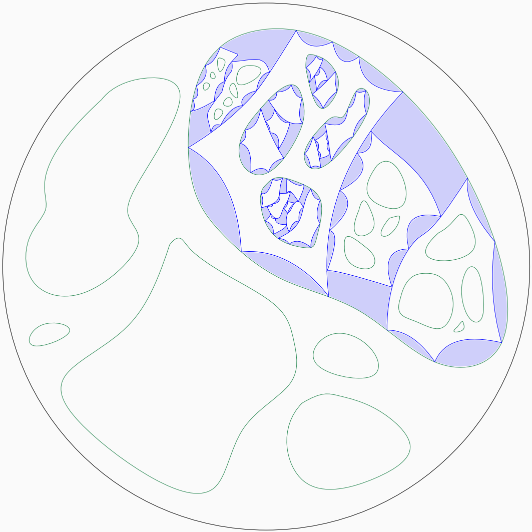
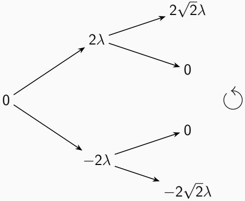


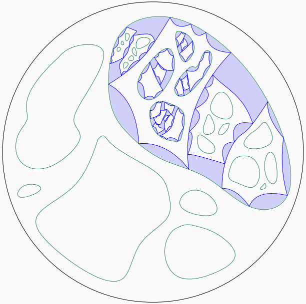


XOR-Ising decomposition

Theorem [AL, Sepúlveda 2026]: The field

$$\tau = \sum_{k=1}^{\infty} \varepsilon_k \nu_k$$

is the scaling limit of the critical XOR-Ising model with free boundary conditions.



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In particular, it has the law of $:\sin((1/\sqrt{2})h):$.

Some remarks

- The signs $(\xi_k)_{k\geq 1}$ used to construct σ are independent of h.

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- But the signs $(\varepsilon_k)_{k\geq 1}$ used to construct τ are functions of h.
- The sums converge under any ordering chosen independently of the signs (but the cancellations are vital).
- Simultaneously, [Aru, Lupu 2026] computed the two-point function of σ directly from the GFF!

The Models

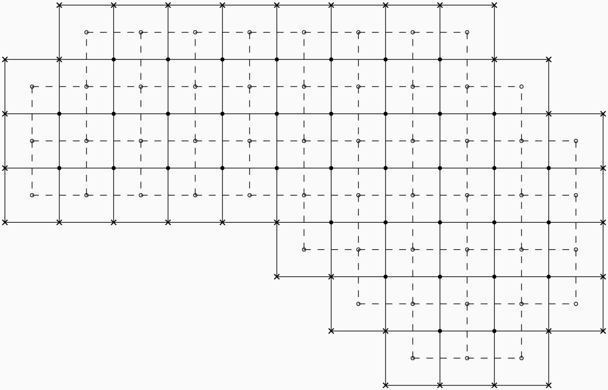
Ising model

Let $G = (V, E) \subset \mathbb{Z}^2$. The Ising model with $+$ boundary conditions and inverse temperature $\beta > 0$ is

$$\mathbb{P}_\beta^+(\sigma) := \frac{1}{Z_\beta^+} \exp\Big\{\beta \sum_{xy \in E} \sigma(x)\sigma(y)\Big\} \mathbf{1}\{\sigma_{|\partial G} \equiv 1\}.$$

Ising model

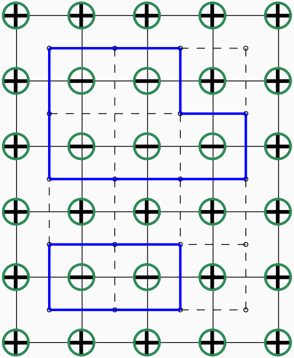
Let $G^\dagger$ be the *weak* dual graph of G.



Ising model

Let $G^\dagger$ be the *weak* dual graph of G.

There is a bijection between $\sigma \in \{\pm 1\}^V$ and $\partial\sigma \in \mathcal{E}(G^\dagger)$.



Every $e^\dagger \in \partial\sigma$ has a cost of $e^{-2\beta}$, hence

$$\mathbb{P}^+_\beta(\sigma) \propto (e^{-2\beta})^{|\partial\sigma|}.$$

Define the dual temperature $\beta^\dagger$ by

$$e^{-2\beta} = \tanh(\beta^\dagger).$$

Ising model

Theorem [Chelkak, Hongler, Izyurov & Camia, Garban, Newman 2015]:
Let $D \subset \mathbb{C}$ be approximated by $D_\delta \subset \delta\mathbb{Z}^2$. At the *critical* temperature β_c,

$$\sigma_\delta := \delta^{2-1/8} \sum_{x \in D_\delta} \sigma(x) \boldsymbol{\delta}_x$$

converges in law as $\delta \to 0$.

There is an alternative, geometric proof based on the FK-Ising representation. Extended by [Camia, Conijn, Kiss 2015] to give a representation in terms of area measures supported on $\mathrm{CLE}_{16/3}$.

XOR-Ising model

Let $G^\dagger$ be the weak dual graph of G. The XOR-Ising model with free boundary conditions and inverse temperature $\beta^\dagger$ is the product

$$\tau^\dagger = \sigma^\dagger \tilde{\sigma}^\dagger$$

of two *independent* Ising models.

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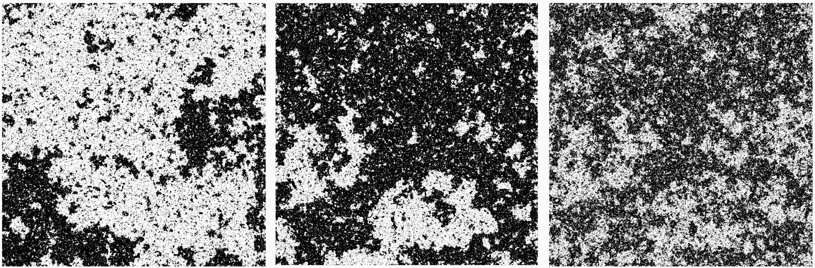
of two *independent* Ising models.

Theorem [Junnila, Saksman, Webb 2020]: At the *critical* temperature β_c,

$$\tau_\delta^\dagger \longrightarrow :\sin((1/\sqrt{2})h):.$$

XOR-Ising model



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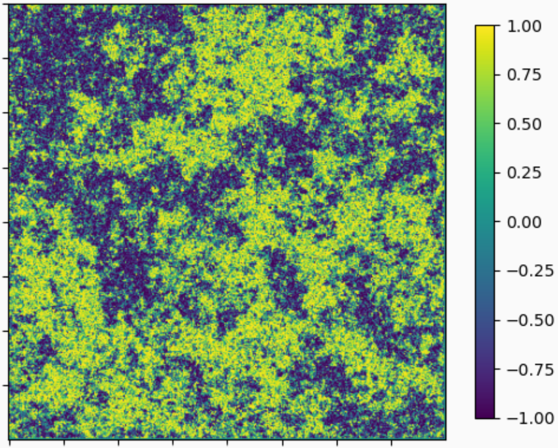
XOR-Ising model



Double random current (DRC)

The (sourceless) random current on $G^\dagger$ with free boundary conditions at inverse temperature $\beta^\dagger$ is the probability measure

$$\mathbf{P}^{\varnothing}_{\beta^\dagger}(\mathbf{n}^\dagger) := \frac{1}{Z^{\varnothing}_{\beta^\dagger}} \prod_{e \in E} \frac{(\beta^\dagger)^{\mathbf{n}^\dagger(e)}}{\mathbf{n}^\dagger(e)!}$$

on $\mathbf{n}^\dagger : E \to \{0, 1, 2, \ldots\}$ such that $\mathbf{n}^\dagger_{\text{odd}} \in \mathcal{E}(G^\dagger)$.

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Without loss of information, consider the "parity projection"

$$\mathbf{P}^{\varnothing}_{\beta^\dagger}(\mathbf{n}^\dagger) \propto \tanh(\beta^\dagger)^{|\mathbf{n}^\dagger_{\mathrm{odd}}|}\ (1 - 1/\cosh(\beta^\dagger))^{|\mathbf{n}^\dagger_{\mathrm{even}}|}\ (1/\cosh(\beta^\dagger))^{|E^\dagger \setminus \mathbf{n}^\dagger|}.$$

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The (sourceless) double random current $\mathbf{n}^\dagger$ is the sum of two *independent* random currents $\mathbf{n}^\dagger_1$ and $\mathbf{n}^\dagger_2$.

The Coupling

Double random current and XOR-Ising

The trace $\hat{\mathbf{n}}^\dagger = \{e \in E : \mathbf{n}^\dagger(e) > 0\}$ induces a percolation model on $G^\dagger$.

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By the *switching lemma* [Griffiths, Hurst, Sherman 1970], e.g.

$$\mathbb{E}_{\beta^\dagger}^{\varnothing}[\tau^\dagger(x)\tau^\dagger(y)] = \mathbf{P}_{\beta^\dagger}^{\varnothing,\mathrm{DRC}}[x \overset{\hat{\mathbf{n}}^\dagger}{\longleftrightarrow} y].$$

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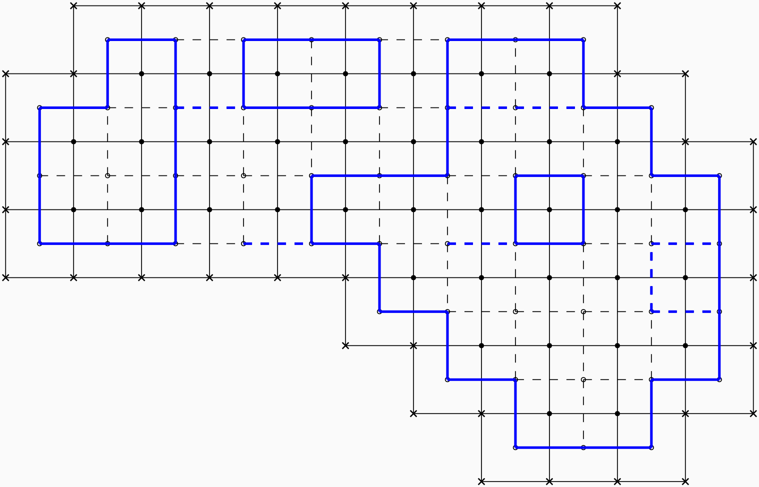
$$\mathbb{E}^{\varnothing}_{\beta^\dagger}[\tau^\dagger(x)\tau^\dagger(y)] = \mathbf{P}^{\varnothing,\mathrm{DRC}}_{\beta^\dagger}[x \xleftrightarrow{\hat{\mathbf{n}}^\dagger} y].$$

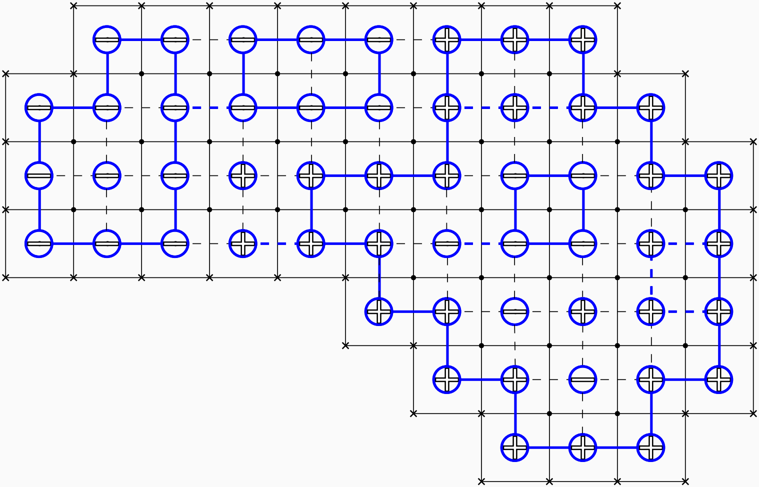
Proposition [Lis 2022 & Duminil-Copin, Lis, Qian 2025]:

$$\tau^\dagger = \sum_{k \geq 1} \varepsilon^\dagger_k \nu^\dagger_k, \quad \text{where} \quad \nu^\dagger_k = \sum_{x \in \mathcal{C}^\dagger_k(\hat{\mathbf{n}}^\dagger)} \boldsymbol{\delta}_x.$$

Building the coupling – Step 1: DRC†



Building the coupling – Step 2: XOR†



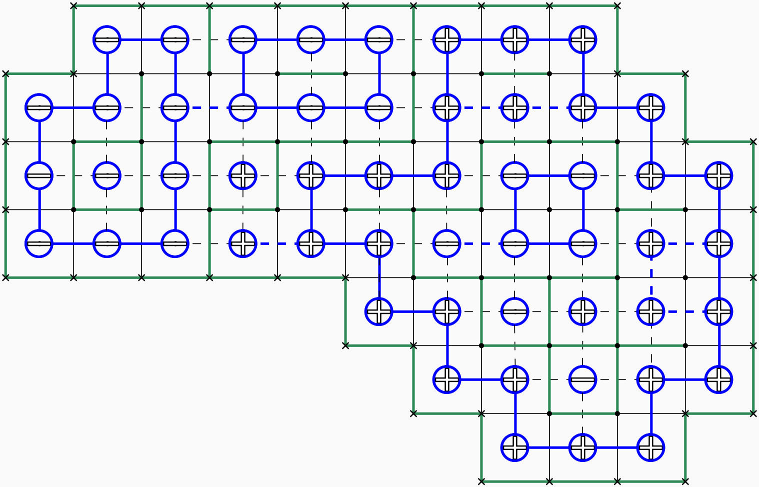
Dual of the double random current and Ising

Theorem [AL, Heeney, Lis 2026]: Let ω be the dual graph of the traced double random current $\hat{\mathbf{n}}^{\dagger}$. Then,

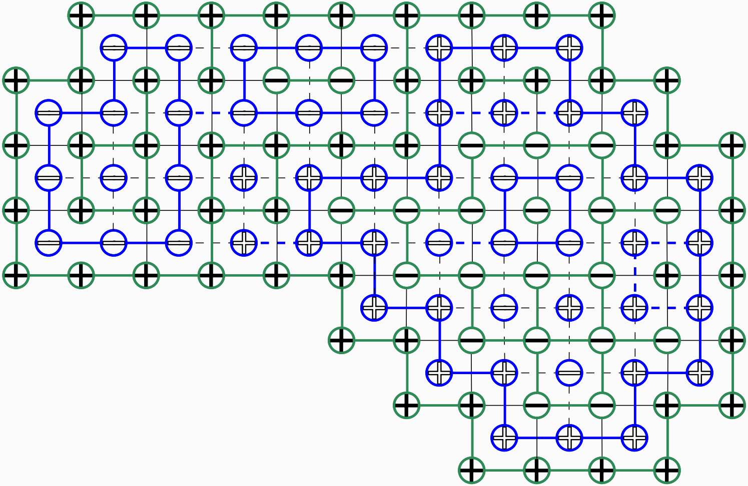
$$\sigma = \mu_0 + \sum_{k \geq 1} \xi_k \mu_k, \quad \text{where} \quad \mu_k = \sum_{x \in \mathcal{C}_k(\omega)} \boldsymbol{\delta}_x.$$

- Percolation representation alternative to the FK-Ising model.
- There are (at least) three different proofs, two hidden in [Lis 2020] and [Lis 2022].

Building the coupling – Step 3: Dual of DRC

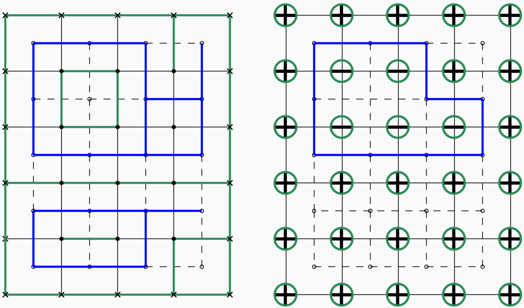


Building the coupling – Step 4: Ising



Dual of the double random current and Ising – Proof Sketch

The statement is equivalent to showing that $\partial\sigma$ is uniform in $\mathcal{E}(\hat{\mathbf{n}}^\dagger)$.



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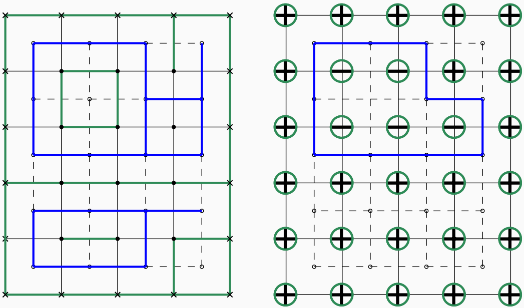

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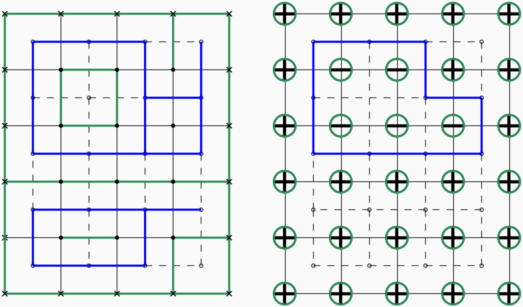

In particular, it is enough to show that $\mathbf{n}^\dagger_{1,\text{odd}}$ is uniform in $\mathcal{E}(\hat{\mathbf{n}}^\dagger)$.

Exercise: Check as multigraphs, then project!

The full coupling

Altogether, there exists a coupling

$$(H, \sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega, \sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger),$$

such that $(\mathbf{n}, \mathbf{n}^\dagger)$ are subdual.

And $(\sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega) \sim (\sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger)$ up to dual boundary conditions.

In fact, each such law does not require planarity to be defined!

[AL, Heeney, Lis 2026]

$$(H, \sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega, \sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger)$$

[Lis 2022]

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[Duminil-Copin, Lis, Qian 2025]

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[Glazmann, Peled 2023]

$$(2H, \sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega, \sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger)$$

[Glazmann, Lammers 2025]

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Main Results – Scaling Limit

Convergence of height function and DRC

Theorem [Duminil-Copin, Lis, Qian 2021-2025]: The joint law

$$(H_\delta, \mathbf{n}_\delta, \mathbf{n}^\dagger_\delta)$$

converges as $\delta \to 0$ to

$$(\frac{1}{\pi\sqrt{2}}h, \mathbf{n}, \mathbf{n}^\dagger).$$

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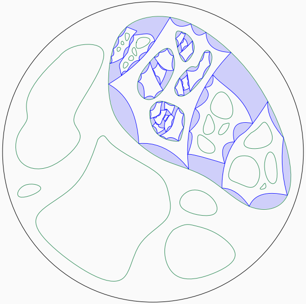

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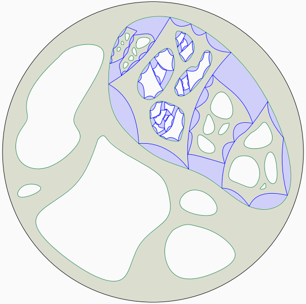
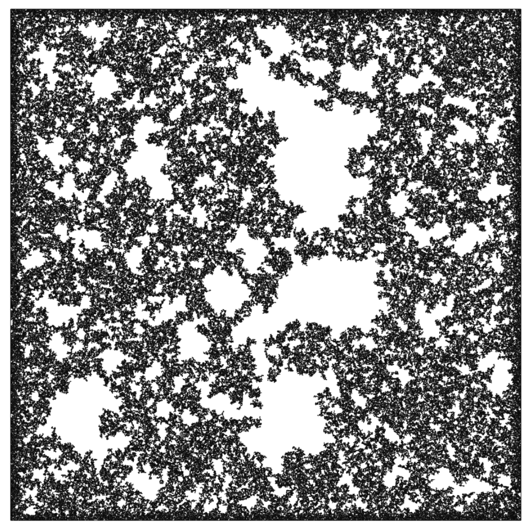


Boundary cluster of ω is $\mathbb{A}_{-2\lambda,2\lambda}$



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Convergence of the full coupling

Theorem [AL, Heeney, Lis 2026 & AL, Sepúlveda 2026]: The joint law

$$(H_\delta, \sigma_\delta, \tilde{\sigma}_\delta, \tau_\delta, \mathbf{n}_\delta, \omega_\delta, \sigma_\delta^\dagger, \tilde{\sigma}_\delta^\dagger, \tau_\delta^\dagger, \mathbf{n}_\delta^\dagger, \omega_\delta^\dagger)$$

converges as $\delta \to 0$ to

$$\left(\frac{1}{\pi\sqrt{2}} h,\ \sigma,\ \tilde{\sigma},\ :\cos\left(\frac{1}{\sqrt{2}} h\right):,\ \mathbf{n},\ \omega,\ \sigma^\dagger,\ \tilde{\sigma}^\dagger,\ :\sin\left(\frac{1}{\sqrt{2}} h\right):,\ \mathbf{n}^\dagger, \omega^\dagger\right).$$

Moreover,

$$:\cos\left(\frac{1}{\sqrt{2}} h\right): = \sigma\tilde{\sigma},$$
$$:\sin\left(\frac{1}{\sqrt{2}} h\right): = \sigma^\dagger\tilde{\sigma}^\dagger.$$

Convergence of the full coupling

Theorem [AL, Heeney, Lis 2026 & AL, Sepúlveda 2026]: The joint law

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And all decompositions converge as e.g.

$$\sigma_\delta = \mu_0^\delta + \sum_{k\geq 1} \xi_k \mu_k^\delta \ \longrightarrow\ \sigma = \mu_0 + \sum_{k\geq 1} \xi_k \mu_k,$$

$$\tau_\delta^\dagger = \sum_{k\geq 1} (\varepsilon_k^\delta)^\dagger (\nu_k^\delta)^\dagger \ \longrightarrow :\sin((1/\sqrt{2})h): = \sum_{k\geq 1} \varepsilon_k^\dagger \nu_k^\dagger.$$

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Some remarks

- One recovers a purely continuum result: all fields can be constructed (decomposed) in terms of a *single* GFF.
- Probabilistic coupling extending the bosonization framework in physics: XOR is a local vertex operator, Ising is a twist field.
- Using [Aru, Junnila 2021], it follows that one can reconstruct a GFF starting from four Ising models.

Thank you!

Conjectures

The Ashkin–Teller model

The Ashkin–Teller measure is

$$\mathbb{P}_G^{\mathsf{AT}}(\sigma, \tilde{\sigma}) := \frac{1}{Z_G^{\mathsf{AT}}} \exp\Big\{ \sum_{xy \in E} \beta\sigma(x)\sigma(y) + \beta\tilde{\sigma}(x)\tilde{\sigma}(y) + U\tau(x)\tau(y) \Big\}$$

The critical line satifies $\sinh(2\beta) = e^{-2U}$ with $U \leq \beta$.

In [Lis 2020] a measure on currents playing the same role of the DRC is defined (and the key switching lemma proved).

Again, some marginals appear in [Glazmann, Peled 2023].

The Ashkin–Teller model

- Based on the mapping to the six-vertex model, it was conjectured (e.g. [Nienhuis 1987]) that the height funtion H converges to

$$\frac{\alpha}{\pi} h \quad \text{for} \quad \alpha = \alpha(U) \in [1/2, \sqrt{3}/2) \approx [1/2, 0.87].$$

Recently proved in infinite volume by [Duminil-Copin, Kozlowski, Lammers, Manolescu 2026] for $\alpha \in [1/2, \sqrt{6}/4] \approx [1/2, 0.61]$.

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- Correspondingly conjectured (e.g. [Kadanoff, Brown 1971]) that

$$\tau \to :\cos(\alpha h):, \quad \tau^\dagger \to :\sin(\alpha h): .$$

The Ashkin–Teller model – New conjectures

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- The Ashkin–Teller currents converge to the iteration of two-valued sets with the appropriate α-dependent gap.
- The (decomposition of the) Ashkin–Teller magnetisation field converges to the appropriate continuum decomposition. And the critical exponent is constant along the whole line!
- The decomposition of the Ashkin–Teller polarisation field converges to the appropriate continuum decomposition.

Near-critical Ising and sine-Gordon

- By [Park 2023], near-critical $\beta = \beta_c + m\delta/2$ Ising correlations converge.
- By [Park, Virtanen, Webb 2025], the near-critical XOR-Ising correlations are those of $:\cos((1/\sqrt{2})\varphi):$ under the free-fermion sine-Gordon model

$$\mathbb{P}^{\mathsf{SG}}_{\left(\beta=\sqrt{2},\ \mu=\frac{-4m}{\pi}\right)}(d\varphi) \propto \exp\left(\mu \int_D :\cos(\beta\phi):\right) \mathbb{P}^{\mathsf{GFF}}(d\varphi).$$

- The complete, analogous discrete coupling holds where the height function is that of *massive* dimers.
- By [Berestycki, Mason, Rey 2026], this height function converges to sine-Gordon *under Temperleyan boundary conditions*.
- The near-critical bosonization coupling should simply change the underlying GFF measure to sine-Gordon!
- In particular, **the (limit of) the near-critical Ising field should be measurable with respect to sine-Gordon and coin tosses**.

Continuum percolations

In the same spirit of [Miller, Sheffield, Werner 2017], what are the continuum percolation procedures to move between

- CLE_3 (Ising interfaces) $\longleftrightarrow$ $\text{CLE}_{16/3}$ (FK-Ising clusters)

  $\longleftrightarrow$ TVS (Dual of DRC clusters)?.

- XOR-Ising interfaces $\longleftrightarrow$ TVS (DRC clusters)?

Thank you!