The (XOR-)Ising model and the Gaussian free field

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Based on joint works with Lorca Heeney, Marcin Lis and Avelio Sepúlveda
(arXiv:2602.05886 & arXiv:2602.06011)

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Vienna Probability Seminar

Outline

1. **Main Results – Continuum**
2. **The Models**
3. **The Coupling**
4. **Main Results – Scaling Limit**
5. **Proof Ideas**
6. **Conjectures – Ashkin-Teller and sine-Gordon**

Main Results – Continuum

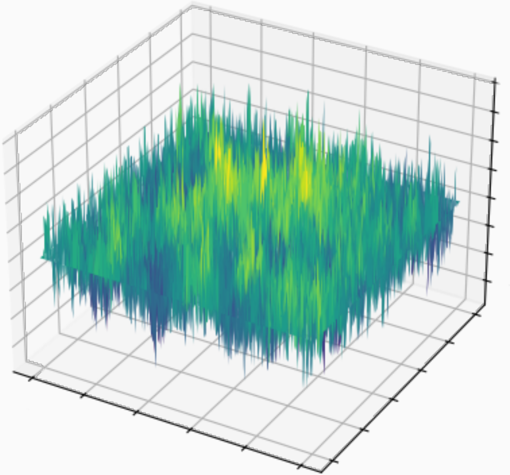
Gaussian free field

The Gaussian free field (GFF) in $D \subset \mathbb{C}$ is the centred Gaussian process h indexed by test functions $f \in C_c^\infty(D)$ with covariance

$$\mathbb{E}[(h,f)(h,g)] = \int_D \int_D G_D(z,w) f(z) g(w) dz dw,$$

where G_D is the Green's function of $-\Delta$.

Guassian free field

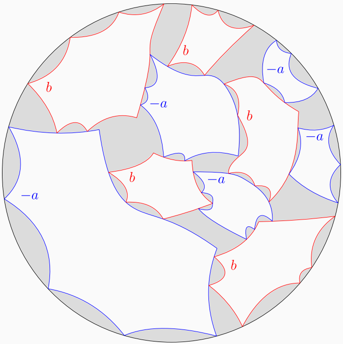


Two-valued sets of the GFF

For $a, b > 0$ with $a + b \geq 2\lambda = \pi$, the two-valued set $\mathbb{A}_{-a,b}$ is the connected component of

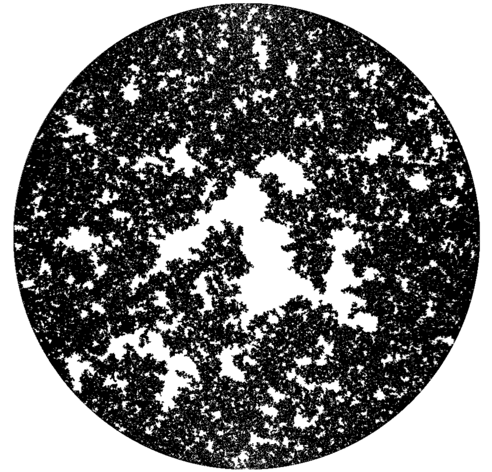
$$\partial D \cup \{z \in D : -a \leq h(z) \leq b\}.$$

Introduced by [Aru, Sepúlveda, Werner 2017].



Two-valued sets of the GFF

Note: The case $a = b = 2\lambda$ is the conformal loop ensemble CLE_4.



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The coupling in one picture

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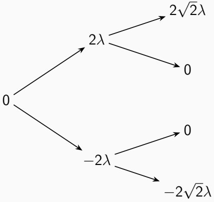

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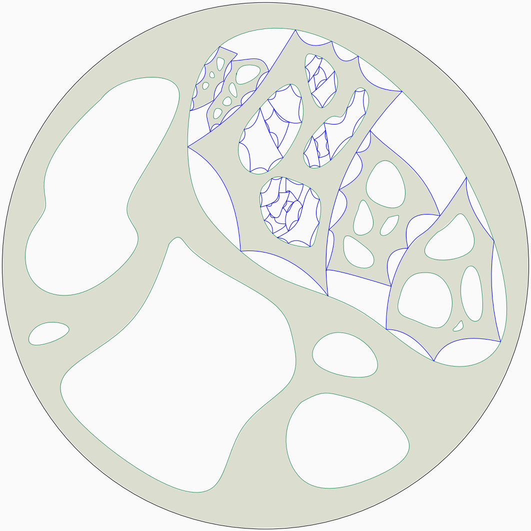
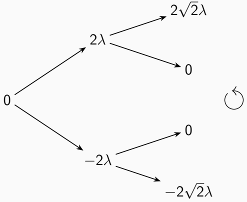


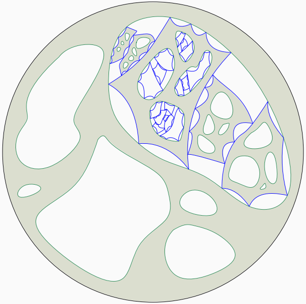


Ising decomposition

Theorem [AL, Heeney, Lis 2026]: The field

$$\sigma = \mu_0 + \sum_{k=1}^{\infty} \xi_k \mu_k$$

is the scaling limit of the critical Ising model with $+$ boundary conditions.



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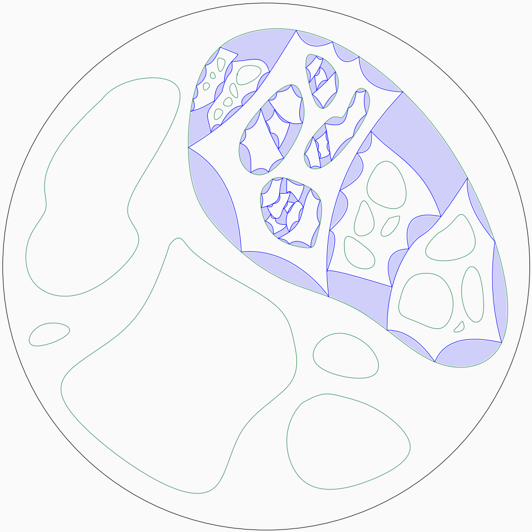
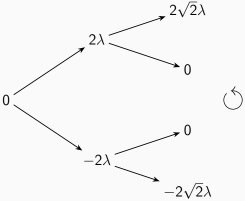


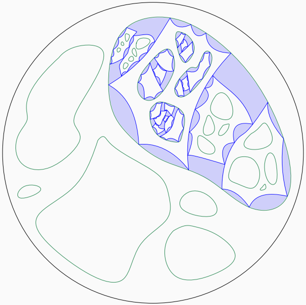


XOR-Ising decomposition

Theorem [AL, Sepúlveda 2026]: The field

$$\tau = \sum_{k=1}^{\infty} \varepsilon_k \nu_k$$

is the scaling limit of the critical XOR-Ising model with free boundary conditions.



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Theorem [AL, Sepúlveda 2026]: The field

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is the scaling limit of the critical XOR-Ising model with free boundary conditions.

In particular, it has the law of $:\sin((1/\sqrt{2})h):$.

Recall: For $\alpha \in (0, \sqrt{2})$,

$$(:e^{i\alpha h}:, f) := \lim_{\varepsilon \to 0} \varepsilon^{-\alpha^2/2} \int_D e^{i\alpha h_\varepsilon(z)} f(z) dz.$$

Some remarks

- The signs $(\xi_k)_{k\geq 1}$ used to construct σ are independent of h.

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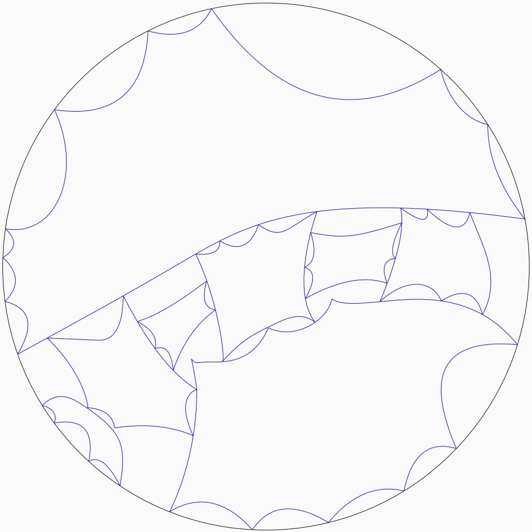
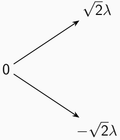
- But the signs $(\varepsilon_k)_{k\geq 1}$ used to construct τ are functions of h.
- The sums converge under any ordering chosen independently of the signs (but the cancellations are vital).

The dual coupling in one picture





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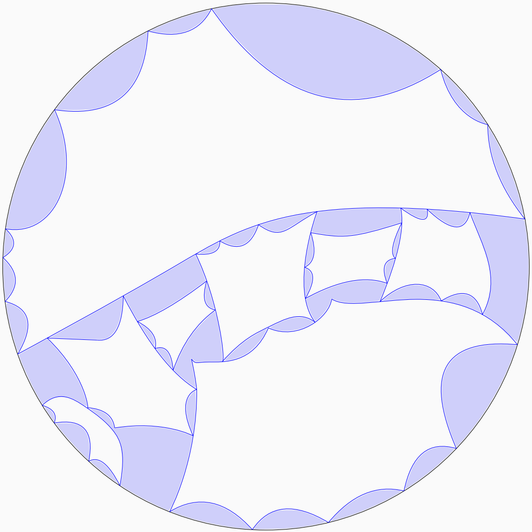
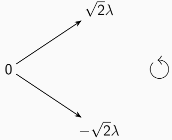




The Models

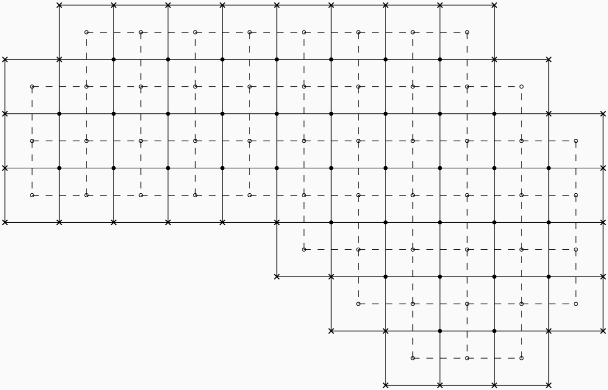
Ising model

Let $G = (V, E) \subset \mathbb{Z}^2$. The Ising model with $+$ boundary conditions and inverse temperature $\beta > 0$ is

$$\mathbb{P}_\beta^+(\sigma) := \frac{1}{Z_\beta^+} \exp\Big\{\beta \sum_{xy \in E} \sigma(x)\sigma(y)\Big\} \mathbf{1}\{\sigma_{|\partial G} \equiv 1\}.$$

Ising model

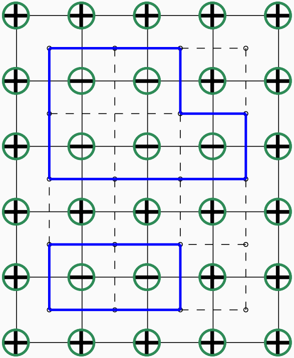
Let $G^\dagger$ be the *weak* dual graph of G.



Ising model

Let $G^\dagger$ be the *weak* dual graph of G.

There is a bijection between $\sigma \in \{\pm 1\}^V$ and $\partial\sigma \in \mathcal{E}_\varnothing(G^\dagger)$.



Every $e^\dagger \in \partial\sigma$ has a cost of $e^{-2\beta}$, hence

$$\mathbb{P}_G^+(\sigma) \propto (e^{-2\beta})^{|\partial\sigma|}.$$

Define the dual temperature $\beta^\dagger$ by

$$e^{-2\beta} = \tanh(\beta^\dagger).$$

Ising model

Theorem [Chelkak, Hongler, Izyurov & Camia, Garban, Newman 2015]:
Let $D \subset \mathbb{C}$ be approximated by $D_\delta \subset \delta\mathbb{Z}^2$. At the *critical* temperature β_c,

$$(\sigma_\delta, f) := \delta^{2-1/8} \sum_{x \in D_\delta} \sigma(x) f(x)$$

converges in law as $\delta \to 0$.

They also provide an alternative, geometric proof based on the FK-Ising representation. Extended by [Camia, Conijn, Kiss 2015] to give a representation in terms of area measures supported on $\mathrm{CLE}_{16/3}$.

XOR-Ising model

Let $G^\dagger$ be the weak dual graph of G. The XOR-Ising model with free boundary conditions and inverse temperature $\beta^\dagger$ is the product

$$\tau^\dagger = \sigma^\dagger \tilde{\sigma}^\dagger$$

of two *independent* Ising models.

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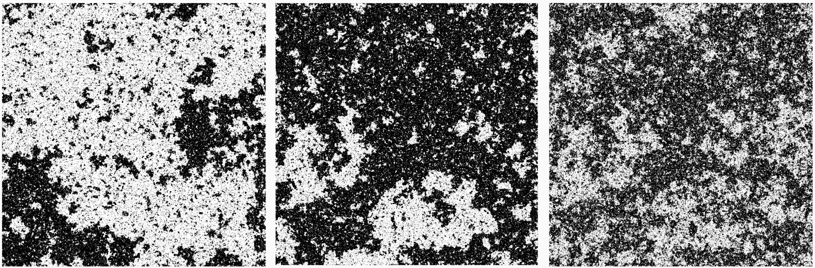
Theorem [Junnila, Saksman, Webb 2020]: At the *critical* temperature β_c,

$$(\tau^\dagger_\delta, f) \longrightarrow (:\sin((1/\sqrt{2})h):, f).$$

Under $+$ boundary conditions,

$$(\tau_\delta, f) \longrightarrow (:\cos((1/\sqrt{2})h):, f).$$

XOR-Ising model



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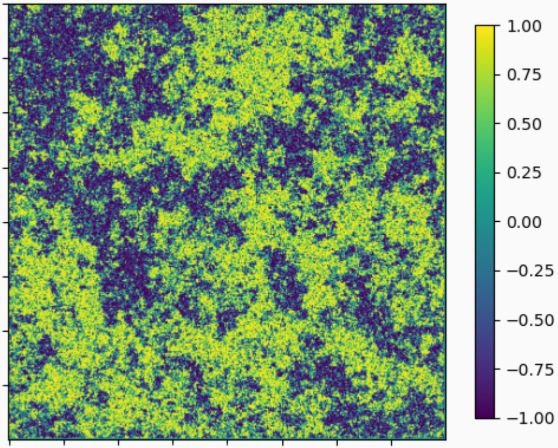
XOR-Ising model



Double random current (DRC)

The (sourceless) random current on $G^\dagger$ with free boundary conditions at inverse temperature $\beta^\dagger$ is the probability measure

$$\mathbf{P}^{\varnothing}_{\beta^\dagger}(\mathbf{n}^\dagger) := \frac{1}{Z^{\varnothing}_{\beta^\dagger}} \prod_{e \in E} \frac{(\beta^\dagger)^{\mathbf{n}^\dagger(e)}}{\mathbf{n}^\dagger(e)!}$$

on $\mathbf{n}^\dagger : E \to \{0, 1, 2, \ldots\}$ such that $\mathbf{n}^\dagger_{\text{odd}} \in \mathcal{E}_\varnothing(G^\dagger)$.

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Without loss of information, consider the "parity projection"

$$\mathbf{P}^{\varnothing}_{\beta^\dagger}(\mathbf{n}^\dagger) \propto \tanh(\beta^\dagger)^{|\mathbf{n}^\dagger_{\text{odd}}|} \, (1 - 1/\cosh(\beta^\dagger))^{|\mathbf{n}^\dagger_{\text{even}}|} \, (1/\cosh(\beta^\dagger))^{|E^\dagger \setminus \mathbf{n}^\dagger|}.$$

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The (sourceless) double random current $\mathbf{n}^\dagger$ is the sum of two *independent* random currents $\mathbf{n}^\dagger_1$ and $\mathbf{n}^\dagger_2$.

The Coupling

Double random current and XOR-Ising

The trace $\hat{\mathbf{n}}^\dagger = \{e \in E : \mathbf{n}^\dagger(e) > 0\}$ induces a percolation model on $G^\dagger$.

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By the *switching lemma* [Griffiths, Hurst, Sherman 1970], e.g.

$$\mathbb{E}^{\varnothing}_{G^\dagger}[\tau^\dagger(x)\tau^\dagger(y)] = \mathbf{P}^{\varnothing,\mathrm{DRC}}_{G^\dagger}[x \xleftrightarrow{\hat{\mathbf{n}}^\dagger} y].$$

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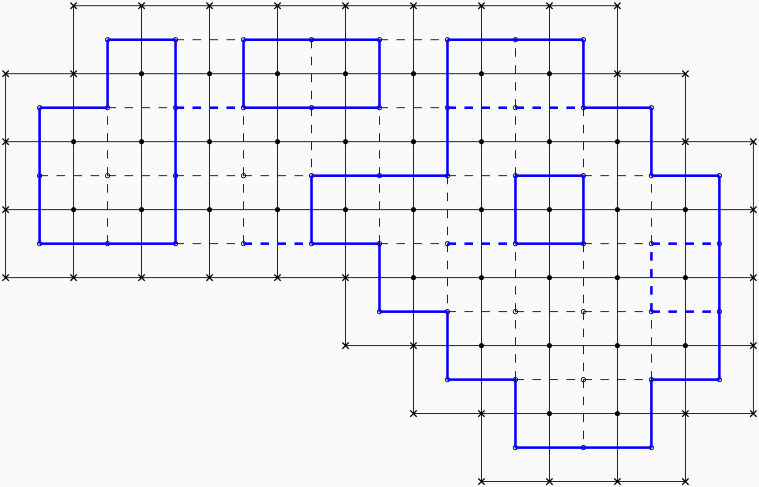
$$\mathbb{E}^{\varnothing}_{G^\dagger}[\tau^\dagger(x)\tau^\dagger(y)] = \mathbf{P}^{\varnothing,\mathrm{DRC}}_{G^\dagger}[x \overset{\hat{\mathbf{n}}^\dagger}{\longleftrightarrow} y].$$

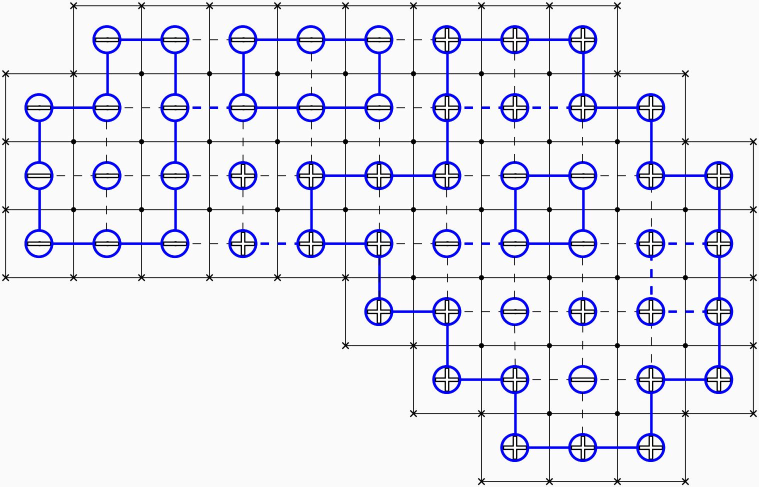
Proposition [Lis 2022 & Duminil-Copin, Lis, Qian 2025]:

$$\tau^\dagger = \sum_{k \geq 1} \varepsilon^\dagger_k \nu^\dagger_k, \quad \text{where} \quad \nu^\dagger_k = \sum_{x \in \mathcal{C}^\dagger_k(\hat{\mathbf{n}}^\dagger)} \boldsymbol{\delta}_x.$$

Building the coupling – Step 1: DRC†



Building the coupling – Step 2: XOR†



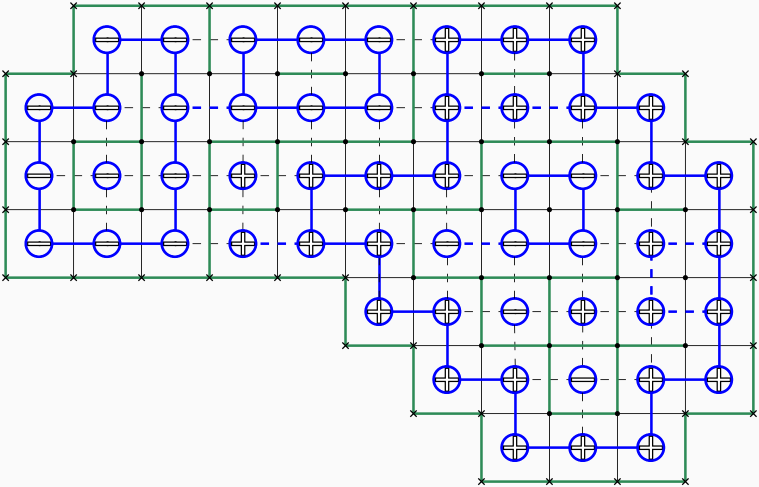
Dual of the double random current and Ising

Theorem [AL, Heeney, Lis 2026]: Let ω be the dual graph of the traced double random current $\hat{\mathbf{n}}^\dagger$. Then,

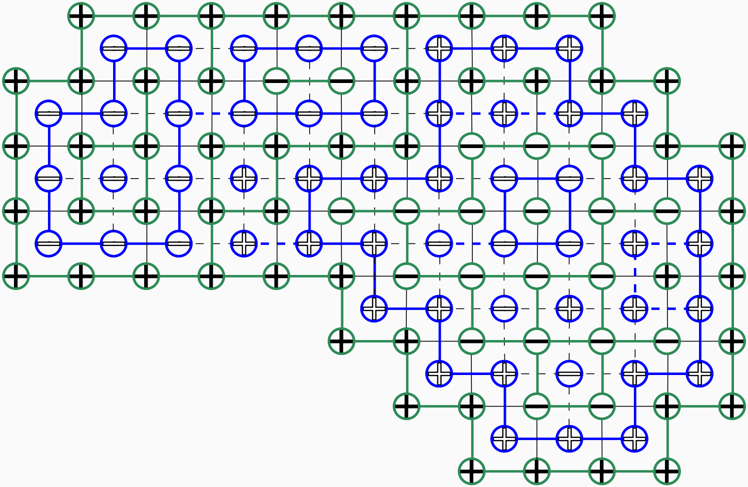
$$\sigma = \mu_0 + \sum_{k\geq 1} \xi_k \mu_k, \quad \text{where} \quad \mu_k = \sum_{x\in\mathcal{C}_k(\omega)} \boldsymbol{\delta}_x.$$

- Percolation representation alternative to the FK-Ising model.
- There are (at least) three different proofs, two hidden in [Lis 2020] and [Lis 2022].

Building the coupling – Step 3: Dual of DRC

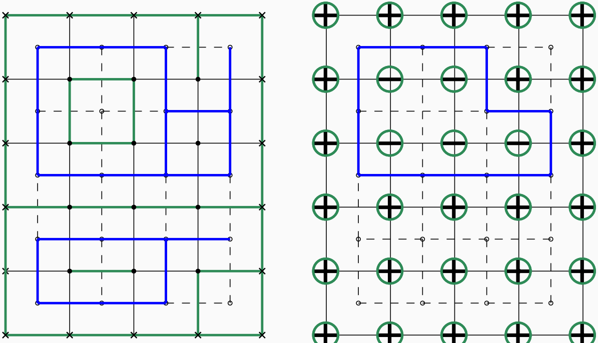


Building the coupling – Step 4: Ising



Dual of the double random current and Ising – Proof

The statement is equivalent to showing that $\partial\sigma$ is uniform in $\mathcal{E}_{\varnothing}(\hat{\mathbf{n}}^{\dagger})$.



Dual of the double random current and Ising – Proof

The statement is equivalent to showing that $\partial\sigma$ is uniform in $\mathcal{E}_{\varnothing}(\hat{\mathbf{n}}^\dagger)$.

In particular, it is enough to show that $\mathbf{n}^\dagger_{1,\mathrm{odd}}$ is uniform in $\mathcal{E}_{\varnothing}(\hat{\mathbf{n}}^\dagger)$.

Claim 1 (Switching lemma): As multigraphs, $\mathbf{n}^\dagger_1$ is uniform in $\mathcal{E}_{\varnothing}(\mathbf{n}^\dagger)$.

Claim 2 (Group homomorphism): The projection

$$\pi : \mathcal{E}_{\varnothing}(\mathbf{n}^\dagger) \to \mathcal{E}_{\varnothing}(\hat{\mathbf{n}}^\dagger) : \mathbf{n}^\dagger_1 \mapsto \mathbf{n}^\dagger_{1,\mathrm{odd}}$$

preserves uniformity.

The height function

The odd part of the DRC $\mathbf{n}^\dagger$ defines the interfaces of a XOR-Ising model τ with $+$ boundary conditions on G.

And $\tilde{\sigma} = \sigma\tau$ must be an Ising model independent of σ.

The height function

The odd part of the DRC $\mathbf{n}^\dagger$ defines the interfaces of a XOR-Ising model τ with $+$ boundary conditions on G.

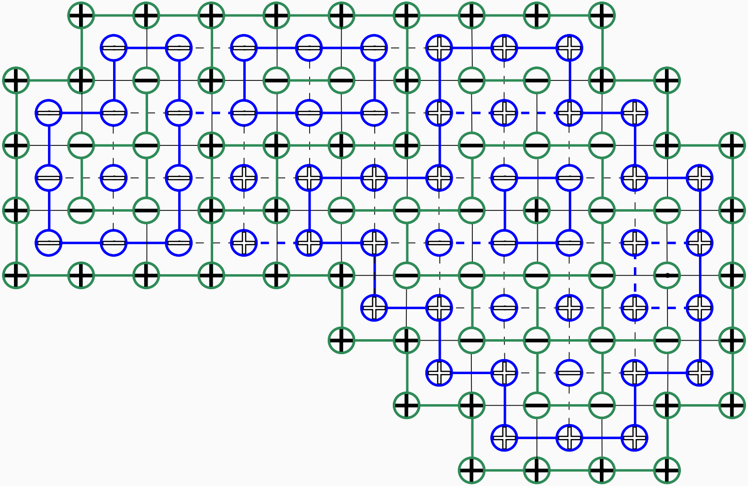
And $\tilde{\sigma} = \sigma\tau$ must be an Ising model independent of σ.

Define a height function H on $V \cup V^\dagger$ by

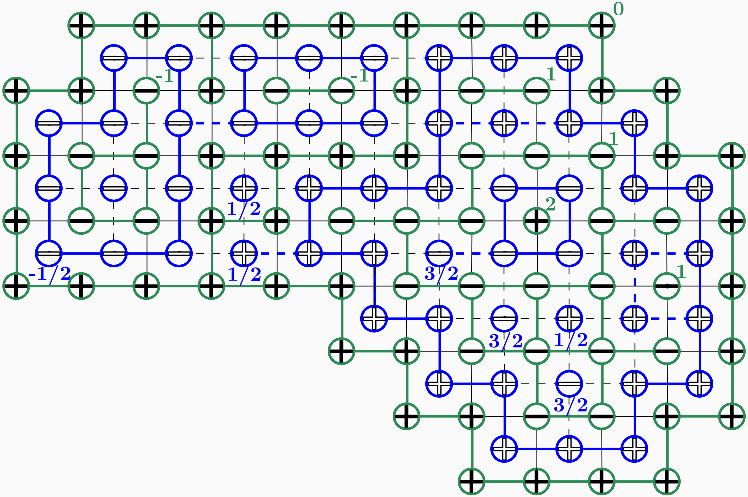
$$H(x^\dagger) - H(x) = \frac{1}{2}\tau_x\tau^\dagger_{x^\dagger} \quad \text{for } x \sim x^\dagger.$$

In particular, $\tau(x) = \cos(\pi H(x))$ and $\tau^\dagger(x^\dagger) = \sin(\pi H(x^\dagger))$.

Building the coupling – Step 5: XOR



Building the coupling – Step 6: Height



The full coupling

Altogether, there exists a coupling

$$(H, \sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega, \sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger).$$

Here $(\mathbf{n}, \mathbf{n}^\dagger)$ are subdual and $(\omega, \omega^\dagger)$ are superdual.

Moreover, $(\sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega)$ and $(\sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger)$ have the same law up to dual boundary conditions.

In fact, each such law does not require planarity to be defined!

[AL, Heeney, Lis 2026]

$$(H, \sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega, \sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger)$$

[Lis 2022]

$$(H, \sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega, \sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger).$$

[Duminil-Copin, Lis, Qian 2025]

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[Glazmann, Peled 2023]

$$(2H, \sigma, \tilde{\sigma}, \tau, \mathbf{n}, \omega, \sigma^\dagger, \tilde{\sigma}^\dagger, \tau^\dagger, \mathbf{n}^\dagger, \omega^\dagger).$$

[Glazmann, Lammers 2025]

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Main Results – Scaling Limit

Convergence of height function and DRC

Theorem [Duminil-Copin, Lis, Qian 2021-2025]: The joint law

$$(H_\delta, \mathbf{n}_\delta, \mathbf{n}^\dagger_\delta)$$

converges as $\delta \to 0$ to

$$(\frac{1}{\pi\sqrt{2}}h, \mathbf{n}, \mathbf{n}^\dagger).$$

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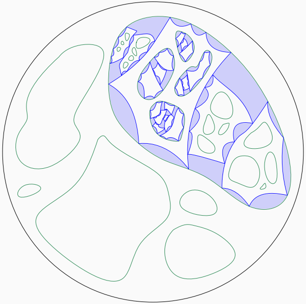

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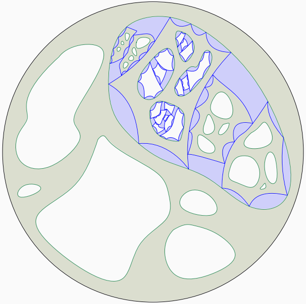

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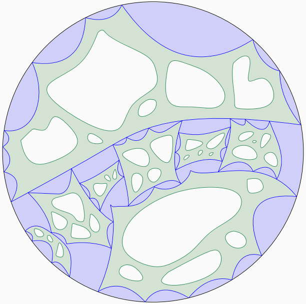
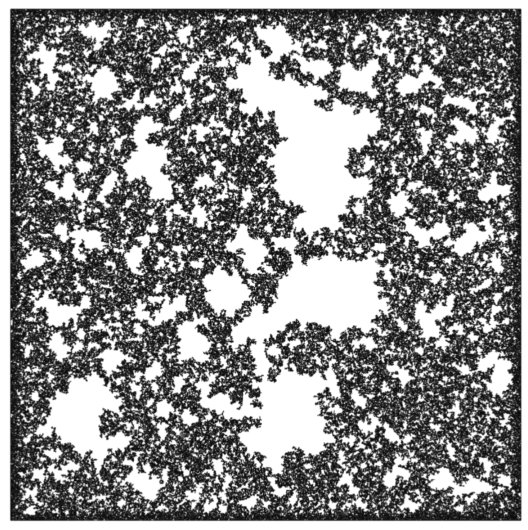


Boundary cluster of ω is $\mathbb{A}_{-2\lambda,2\lambda}$



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Passing to the limit

Question: Can one take the scaling limit $\delta \to 0$ of the *complete* coupling? How much of it survives? In particular, do the representations in terms of area measures survive?

Passing to the limit

Question: Can one take the scaling limit $\delta \to 0$ of the *complete* coupling? How much of it survives? In particular, do the representations in terms of area measures survive?

Answer: Yes. Everything*. Yes.

* Except interfaces...

Convergence of the full coupling

Theorem [AL, Heeney, Lis 2026 & AL, Sepúlveda 2026]: The joint law

$$(H_\delta, \sigma_\delta, \tilde{\sigma}_\delta, \tau_\delta, \mathbf{n}_\delta, \omega_\delta, \sigma_\delta^\dagger, \tilde{\sigma}_\delta^\dagger, \tau_\delta^\dagger, \mathbf{n}_\delta^\dagger, \omega_\delta^\dagger)$$

converges as $\delta \to 0$ to

$$\left(\frac{1}{\pi\sqrt{2}}h,\ \sigma,\ \tilde{\sigma},\ :\cos\left(\frac{1}{\sqrt{2}}h\right):,\ \mathbf{n},\ \omega,\ \sigma^\dagger,\ \tilde{\sigma}^\dagger,\ :\sin\left(\frac{1}{\sqrt{2}}h\right):,\ \mathbf{n}^\dagger, \omega^\dagger\right).$$

Moreover,

$$:\cos\left(\frac{1}{\sqrt{2}}h\right): = \sigma\tilde{\sigma},$$
$$:\sin\left(\frac{1}{\sqrt{2}}h\right): = \sigma^\dagger\tilde{\sigma}^\dagger.$$

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$$\left(\frac{1}{\pi\sqrt{2}}h,\ \sigma,\ \tilde{\sigma},\ :\cos\left(\frac{1}{\sqrt{2}}h\right):,\ \mathbf{n},\ \omega,\ \sigma^\dagger,\ \tilde{\sigma}^\dagger,\ :\sin\left(\frac{1}{\sqrt{2}}h\right):,\ \mathbf{n}^\dagger, \omega^\dagger\right).$$

And all decompositions converge as e.g.

$$\sigma_\delta = \mu_0^\delta + \sum_{k\geq 1} \xi_k \mu_k^\delta \ \longrightarrow\ \sigma = \mu_0 + \sum_{k\geq 1} \xi_k \mu_k,$$

$$\tau_\delta^\dagger = \sum_{k\geq 1} (\varepsilon_k^\delta)^\dagger (\nu_k^\delta)^\dagger \ \longrightarrow :\sin((1/\sqrt{2})h): = \sum_{k\geq 1} \varepsilon_k^\dagger \nu_k^\dagger.$$

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- Based on this twist field description, Aru and Lupu independently conjectured the coupling of an Ising field with the GFF, and compute the Ising two-point function directly from the GFF.
- Using [Aru, Junnila 2021], it follows that one can reconstruct a GFF starting from four Ising models!

Decomposition of complex chaos

Theorem [AL, Sepúlveda 2026]: For any $\alpha \in (0,1)$,

$$:\sin(\alpha h): = \sum_{k\geq 1} \varepsilon_k^\dagger \nu_k^\dagger,$$

$$:\cos(\alpha h): = \nu_0 + \sum_{k\geq 1} \varepsilon_k \nu_k,$$

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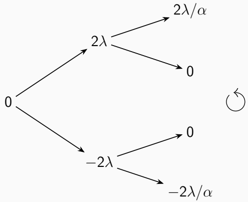

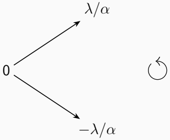

Thank you!

Proof Ideas

Proof ideas – Ising

Key claim: For any cluster $\mathcal{C}_\delta$ of ω_δ, $(\mathcal{C}_\delta, \mu_\delta) \longrightarrow (\mathcal{C}, \mu)$ with $\mu = \mu(\mathcal{C})$.

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Following [Garban, Pete, Schramm 2013], which was also used by [CGN15] and [CCK15], we show that as $\delta \to 0$ and then $\varepsilon \to 0$,

$$\mathbb{E}\left[\left(\mu_\delta[D] - (\delta/\varepsilon)^{2-1/8} \sum_{\varepsilon\text{-boxes } B} \mathbf{1}\{B \cap \mathcal{C}_\delta\} \mathbb{E}\big[|B \cap \mathcal{C}_\delta| \big| B \cap \mathcal{C}_\delta\big]\right)^2\right] \longrightarrow 0.$$

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This approximation requires many tools:

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- **Construction of IIC measure**: Infinite volume limit "conditioned" on the existence of an infinite cluster.

Proof ideas – XOR

Build on the relation between complex chaos and two-valued sets introduced in [Schoug, Sepúlveda, Viklund 2022].

For $\alpha \in (0,1)$, let $a_c = a_c(\alpha) = \lambda/\alpha$ so that

$$\dim_H(\mathbb{A}_{-a_c,a_c}) = 2 - \frac{\alpha^2}{2}.$$

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The last two properties rely on the fact that

$$\mathbb{E}[:\sin(\alpha\phi(z))::\sin(\alpha\phi(w)):]$$

is increasing in the domain: uses Loewner(-Kufarev) chains!

Conjectures

The Ashkin–Teller model

The Ashkin–Teller measure is

$$\mathbb{P}_G^{\mathsf{AT}}(\sigma) := \frac{1}{Z_G^{\mathsf{AT}}} \exp\Big\{ \sum_{xy \in E} \beta\sigma(x)\sigma(y) + \beta\tilde{\sigma}(x)\tilde{\sigma}(y) + U\tau(x)\tilde{\tau}(y) \Big\}$$

The critical line satifies $\sinh(2\beta) = e^{-2U}$ with $U \leq \beta$.

In [Lis 2020] a measure on currents playing the same role of the DRC is defined (and the key switching lemma proved).

Again, some marginals appear in [Glazmann, Peled 2023].

The Ashkin–Teller model

- Based on the mapping to the six-vertex model, it was conjectured (e.g. [Nienhuis 1987]) that the height funtion H converges to

$$\frac{\alpha}{\pi} h \quad \text{for} \quad \alpha = \alpha(U) \in [1/2, \sqrt{3}/2) \approx [1/2, 0.87].$$

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- Correspondingly conjectured (e.g. [Kadanoff, Brown 1971]) that

$$\tau \to :\cos(\alpha h):, \quad \tau^{\dagger} \to :\sin(\alpha h): .$$

The Ashkin–Teller model – New conjectures

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- The (decomposition of the) Ashkin–Teller magnetisation field converges to the appropriate continuum decomposition. And the critical exponent is constant along the whole line!
- The decomposition of the Ashkin–Teller polarisation field converges to the appropriate continuum decomposition.

Near-critical Ising and sine-Gordon

The near-critical scaling limit $\beta = \beta_c - m\delta/2$, $m < 0$ of Ising correlation functions was established in [Park 2023].

The near-critical XOR-Ising correlations are [Park, Virtanen, Webb 2025] those of $:\cos((1/\sqrt{2})\varphi):$ under the free-fermion sine-Gordon model

$$\mathbb{P}^{\mathsf{SG}}_{\left(\beta=\sqrt{2},\mu=\frac{-4m}{\pi}\right)}(d\varphi) \propto \exp\left(\mu \int_D :\cos(\beta\phi):\right) \mathbb{P}^{\mathsf{GFF}}(d\varphi).$$

The complete, analogous discrete coupling is as above, but now the height function is that of *massive* dimers. Under Temperleyan boundary conditions, convergence of this height function to sine-Gordon was proved in [Berestycki, Mason, Rey 2026].

Upon changing underlying measure from GFF to sine-Gordon, many aspects should be the same! In particular, **the scaling limit of the near-critical Ising field should be measurable with respect to sine-Gordon and additional coin tosses**.

Continuum percolations

In the same spirit of [Miller, Sheffield, Werner 2017], what are the continuum percolation procedures to move between

- CLE_3 (Ising interfaces) $\longleftrightarrow$ $\text{CLE}_{16/3}$ (FK-Ising clusters)

  $\longleftrightarrow$ TVS (Dual of DRC clusters)?.

- XOR-Ising interfaces $\longleftrightarrow$ TVS (DRC clusters)?

Thank you!